

THE COMPLETE CHARACTERIZATION OF A.S. CONVERGENCE OF ORTHOGONAL SERIES

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In this paper we prove the complete characterization of a.s. convergence of orthogonal series in terms of existence of a majorizing measure. It means that for a given $(a_n)_{n=1}^\infty$, $a_n > 0$, series $\sum_{n=1}^\infty a_n \varphi_n$ is a.e. convergent for each orthonormal sequence $(\varphi_n)_{n=1}^\infty$ if and only if there exists a measure m on

$$T = \{0\} \cup \left\{ \sum_{n=1}^m a_n^2, m \geq 1 \right\}$$

such that

$$\sup_{t \in T} \int_0^{\sqrt{D(T)}} (m(B(t, r^2)))^{-1/2} dr < \infty,$$

where $D(T) = \sup_{s, t \in T} |s - t|$ and $B(t, r) = \{s \in T : |s - t| \leq r\}$. The presented approach is based on weakly majorizing measures and a certain partitioning scheme.

1. Introduction. An orthonormal sequence $(\varphi_n)_{n=1}^\infty$ on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is a sequence of random variables $\varphi_n : \Omega \rightarrow \mathbb{R}$ such that $\mathbf{E}\varphi_n^2 = 1$ and $\mathbf{E}\varphi_n \varphi_m = 0$ whenever $n \neq m$. In this paper we consider the question of how to characterize the sequences of $(a_n)_{n=1}^\infty$ for which the series

$$\sum_{n=1}^\infty a_n \varphi_n \text{ converges a.e. for any orthonormal } (\varphi_n)_{n=1}^\infty$$

on any probability spaces $(\Omega, \mathcal{F}, \mathbf{P})$. Note that we can assume $a_n > 0$ for $n \geq 1$. The answer is based on the analysis of the set

$$T = \left\{ \sum_{n=1}^m a_n^2 : m \geq 1 \right\} \cup \{0\}.$$

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The classical Rademacher–Menchof theorem (see [4, 5]) states that $\sum_{n=1}^{\infty} a_n^2 \times \log^2(n+1)$ suffices for $\sum_{n=1}^{\infty} a_n \varphi_n$ convergence. Another well-known observation (see [12]) is the following theorem.

THEOREM 1. *For each orthonormal sequence $(\varphi_n)_{n=1}^{\infty}$ the series $\sum_{n=1}^{\infty} a_n \varphi_n$ converges a.e. if and only if*

$$\mathbf{E} \sup_{m \geq 1} \left(\sum_{n=1}^m a_n \varphi_n \right)^2 < \infty.$$

The consequence of the above result is that the main problem can be reformulated in terms of sample boundedness of all orthogonal processes on T . We say that process $X(t)$, $t \in T$, is of orthogonal increments if

$$(1) \quad \mathbf{E}(X(s) - X(t))^2 = |s - t| \quad \text{for } s, t \in T.$$

There is a bijection between orthonormal series $\sum_{n=1}^{\infty} a_n \varphi_n$ and processes with orthogonal increments on T . Namely for each sequence $(\varphi_n)_{n=1}^{\infty}$ we define processes

$$X(t) = \sum_{n=1}^m a_n \varphi_n \quad \text{for } t = \sum_{n=1}^m a_n^2, X(0) = 0,$$

and for each orthogonal process $X(t)$, $t \in T$, we define the orthonormal sequence by

$$\varphi_m = a_m^{-1} \left(X \left(\sum_{n=1}^m a_n^2 \right) - X \left(\sum_{n=0}^{m-1} a_n^2 \right) \right) \quad \text{for } m > 1,$$

and $\varphi_1 = a_1^{-1}(X(a_1^2) - X(0))$. By Theorem 1, each orthogonal series $\sum_{n=1}^{\infty} a_n \varphi_n$ is a.e. convergent if and only if there exists a universal constant $\mathcal{M} < \infty$ such that

$$(2) \quad \mathbf{E} \sup_{t \in T} |X(t) - X(0)|^2 \leq \mathcal{M}$$

for all processes $X(t)$, $t \in T$ that satisfy (1).

We treat the generalized question and consider any $T \subset \mathbb{R}$. The best tool which is used to study the sample boundedness of orthogonal processes on T are majorizing measures. Let $B(t, r) = \{s \in T : |s - t| \leq r\}$ and $D(T) = \sup_{s, t \in T} |s - t|$. We say that a probability measure m on T is majorizing (in the orthogonal setting) if

$$\sup_{t \in T} \int_0^{\sqrt{D(T)}} (m(B(t, r^2)))^{-1/2} dr < \infty.$$

We say that a process $X(t)$, $t \in T$, is of suborthogonal increments if

$$(3) \quad \mathbf{E}|X(s) - X(t)|^2 \leq |s - t| \quad \text{for } s, t \in T.$$

Corollary 1 proved in [2] states that the existence of a majorizing measure is the necessary and sufficient condition for the sample boundedness of all suborthogonal processes. Moreover by Theorem 3.2 in [1] (see also [10]) we have the following theorem.

THEOREM 2. *For each process $X(t)$, $t \in T$, that satisfies (3), the following inequality holds:*

$$\mathbf{E} \sup_{s, t \in T} |X(s) - X(t)|^2 \leq 16 \cdot 5^{5/2} \left(\sup_{t \in T} \int_0^{\sqrt{D(T)}} (m(B(t, r^2)))^{-1/2} dr \right)^2.$$

Consequently the existence of a majorizing measure is always sufficient for the a.e. convergence of orthogonal series $\sum_{n=1}^{\infty} a_n \varphi_n$.

The problem is that the class of orthogonal processes is significantly smaller than the class of processes that verify (3). Only recently Paszkiewicz proved in [6, 7], using advanced methods of entropy of interval, that the existence of a majorizing measure is also necessary for all orthogonal processes to satisfy (2). This motivated our research for an alternative approach entirely based on the generic chaining; see [8, 11]. We use the Fernique's idea of constructing a majorizing measure. We say that a probability measure μ on T is weakly majorizing if

$$\int_T \int_0^{\sqrt{D(T)}} (\mu(B(t, r^2)))^{-1/2} dr \mu(dt) < \infty.$$

Let

$$\mathcal{M} = \sup_{\mu} \int_T \int_0^{\sqrt{D(T)}} (\mu(B(t, r^2)))^{-1/2} dr \mu(dt),$$

where the supremum is taken over all probability measures on T .

THEOREM 3 [3, 10]. *If $\mathcal{M} < \infty$, that is, all probability measures are weakly majorizing with a uniform bounding constant, then there exists m a majorizing measure on T such that*

$$\sup_{t \in T} \int_0^{\sqrt{D(T)}} (m(B(t, r^2)))^{-1/2} dr \leq \mathcal{M}.$$

The main result of this paper is the following theorem.

THEOREM 4. *Whenever all orthogonal processes on T satisfy (2), then $\mathcal{M} \leq KD(T)$, where $K < \infty$.*

When combined with Theorems 1, 3, 4 it implies the complete characterization of a.e. convergence of all orthogonal series.

COROLLARY 1. *For a given $(a_n)_{n=1}^\infty$ series $\sum_{n=1}^\infty a_n \varphi_n$ are a.e. convergent for all orthonormal sequences $(\varphi_n)_{n=1}^\infty$ if and only if there exists a majorizing measure m on T .*

We stress that using the chaining argument and the Fernique's idea of constructing a majorizing measure makes the proof significantly shorter than the one presented in [6].

2. Structure of the proof. If all orthogonal process satisfy (2), then in particular $D(T) < \infty$. For simplicity assume that $T \subset [0, 1)$ (the general result can be obtained by the translation invariance and homogeneity). Our approach is based on proving special properties of natural partitions of $[0, 1)$. Let

$$(4) \quad \mathcal{A}_k = \{A_i^{(k)} : 0 \leq i < 4^k\}, \quad k \geq 0 \text{ where } A_i^{(k)} = [i4^{-k}, (i+1)4^{-k}) \cap T,$$

in particular $A_0^{(0)} = T$. In Section 3 we translate the weakly majorizing measure functionals into the language of \mathcal{A}_k , $k \geq 0$. Since as sated in Theorem 3 we have to deal with any probability measure μ on T , we fix μ and check that for the particular $0 \leq i < 4^k$ sets $A_{4i+j}^{(k)}$, $j \in \{0, 1, 2, 3\}$, are important only if the measure μ of $A_i^{(k-1)}$ is well distributed among them. In this way we obtain the quantity that one may use to bound the weakly majorizing measure functional.

Then we follow the idea that was first invented by Talagrand in [9] to prove the complete characterization of Gaussian sample boundedness. We introduce the set functionals F_k , $k \geq 0$, such that F_k operates on \mathcal{A}_k and is given by

$$(5) \quad F_k(A_i^{(k)}) = \sup_Y \mathbf{E} \sup_{t \in A_i^{(k)}} Y(t),$$

where the supremum is over the class of processes $Y(t)$, $t \in \bar{A}_i^{(k)}$, where $\bar{A}_i^{(k)} = A_i^{(k)} \cup \{i4^{-k}, (i+1)4^{-k}\}$, that satisfy $\mathbf{E}Y(t) = 0$ and

$$(6) \quad \mathbf{E}|Y(s) - Y(t)|^2 = |s - t|(1 - 4^k|s - t|) \quad \text{for } s, t \in \bar{A}_i^{(k)}.$$

In particular $Y(i4^{-k}) = Y((i+1)4^{-k})$, and hence we may require $Y(i4^{-k}) = Y((i+1)4^{-k}) = 0$ [it does not change $F_k(A_i^{(k)})$]. We show in Section 4 that if (2) holds for all orthogonal processes, then $F_0(T) < \infty$. The partitioning scheme is the induction step which shows that partitioning of $A_i^{(k-1)}$

into $A_{4i+j}^{(k)}$, $j \in \{0, 1, 2, 3\}$, makes it possible to earn the suitable quantity so that summing all over the partitions completes the argument of the uniform bound existence for any weakly majorizing measure functional. The proof of the induction step is the construction for a fixed $0 \leq i < 4^{k-1}$ of a special process $Y(t)$, $t \in \bar{A}_i^{k-1}$, that satisfies (6). In the construction we use optimal (or nearly optimal) processes on $A_{4i+j}^{(k)}$ for $j \in \{0, 1, 2, 3\}$ and a suitably chosen family of independent random variables.

3. Weakly majorizing measures. We have noted in Section 2 that one may assume $T \subset [0, 1)$. Consequently μ is weakly majorizing if

$$(7) \quad \int_T \int_0^1 (\mu(B(t, r^2)))^{-1/2} dr \mu(dt) < \infty.$$

We first translate the functional from (7) into the language of \mathcal{A}_k , $k \geq 0$, defined in (4).

LEMMA 1. *For each measure μ the inequality holds*

$$\int_T \int_0^1 (\mu(B(t, r^2)))^{-1/2} dr \mu(dt) \leq \sum_{k=1}^{\infty} 2^{-k} \sum_{i=0}^{4^k-1} (\mu(A_i^{(k)}))^{1/2}.$$

PROOF. First observe that

$$\int_0^1 (\mu(B(t, r^2)))^{-1/2} dr \leq \sum_{k=1}^{\infty} 2^{-k} (\mu(B(t, 4^{-k})))^{-1/2} \quad \text{for } t \in T.$$

Clearly $|\mathcal{A}_k| \leq 4^k$ and $A_i^{(k)} \subset B(t, 4^{-k})$ for all $t \in A_i^{(k)} \in \mathcal{A}_k$. Consequently $\mu(A_i^{(k)}) \leq \mu(B(t, 4^{-k}))$, and hence

$$\begin{aligned} \int_T (\mu(B(t, 4^{-k})))^{-1/2} \mu(dt) &\leq \sum_{i=0}^{4^k-1} \int_{A_i^{(k)}} (\mu(B(t, 4^{-k})))^{-1/2} \mu(dt) \\ &\leq \sum_{i=0}^{4^k-1} \int_{A_i^{(k)}} (\mu(A_i^{(k)}))^{-1/2} \mu(dt) = \sum_{i=0}^{4^k-1} (\mu(A_i^{(k)}))^{1/2}. \end{aligned}$$

Therefore

$$\begin{aligned} \int_T \int_0^1 (\mu(B(t, r)))^{-1/2} dr \mu(dt) &\leq \sum_{k=1}^{\infty} 2^{-k} \int_T (\mu(B(t, 4^{-k})))^{-1/2} \mu(dt) \\ &\leq \sum_{k=1}^{\infty} 2^{-k} \sum_{i=0}^{4^k-1} (\mu(A_i^{(k)}))^{1/2}. \end{aligned}$$

□

For a specific measure μ not all subsets $A_i^{(k)} \in \mathcal{A}_k$ are important. Observe that for $0 \leq i < 4^{k-1}$, $\bigcup_{j=0}^3 A_{4i+j}^{(k)} = A_i^{(k-1)}$. Denote by $I(k)$ the set of indices $4i + j$ where $0 \leq i < 4^k$, $0 \leq j \leq 3$ such that

$$(8) \quad \frac{1}{32}\mu(A_i^{(k-1)}) \leq \mu(A_{4i+j}^{(k)}) \leq \frac{1}{2}\mu(A_{4i}^{(k)} \cup A_{4i+2}^{(k)})$$

if $j \in \{0, 2\}$, and

$$(9) \quad \frac{1}{32}\mu(A_i^{(k-1)}) \leq \mu(A_{4i+j}^{(k)}) \leq \frac{1}{2}\mu(A_{4i+1}^{(k)} \cup A_{4i+3}^{(k)})$$

if $j \in \{1, 3\}$. The meaning of the construction is that $4i + j \in I(k)$ only if measure of $A_i^{(k-1)}$ is well distributed among $A_{4i+j}^{(k)}$, $j \in \{0, 1, 2, 3\}$.

We improve Lemma 1, showing that the upper bound for the weakly majorizing measure functional can be replaced by the one that uses only sets of the form $A_i^{(k)}$, $i \in I(k)$.

PROPOSITION 1. *For each probability Borel measure μ on T , the following inequality holds:*

$$\int_T \int_0^1 (\mu(B(t, \varepsilon)))^{-1/2} \leq \frac{1}{1 - 2^{-1}L} \left[L + \sum_{k=1}^{\infty} 2^{-k} \sum_{i=0}^{4^k-1} (\mu(A_i^{(k)}))^{1/2} 1_{i \in I(k)} \right],$$

where $L = 2^{1/2} \cdot \frac{5}{4} < 2$.

PROOF. Suppose that $4i + j \notin I(k)$ and $j \in \{0, 2\}$, then there are two possibilities, either

$$(10) \quad \mu(A_{4i+j}^{(k)}) < \frac{1}{32}\mu(A_i^{(k-1)}), \quad \text{or}$$

$$(11) \quad \mu(A_{4i}^{(k)} \cup A_{4i+2}^{(k)}) > 2\mu(A_{4i+j}^{(k)}).$$

If (10) holds, then

$$(12) \quad (\mu(A_{4i+j}^{(k)}))^{1/2} < \frac{2^{1/2}}{8}(\mu(A_i^{(k-1)}))^{1/2}.$$

Assuming (11) we use the trivial inequality

$$(13) \quad (\mu(A_i^{(k)}))^{1/2} < (\mu(A_{4i}^{(k)} \cup A_{4i+2}^{(k)}))^{1/2}.$$

One cannot have that both $j = 0$ and $j = 2$ satisfy (11), and therefore due to (12) and (13),

$$(14) \quad \begin{aligned} & (\mu(A_{4i}^{(k)}))^{1/2} 1_{4i \notin I(k)} + (\mu(A_{4i+2}^{(k)}))^{1/2} 1_{4i+2 \notin I(k)} \\ & \leq \max\left\{\frac{2^{1/2}}{4}(\mu(A_i^{(k-1)}))^{1/2}, \frac{2^{1/2}}{8}(\mu(A_i^{(k-1)}))^{1/2} \right. \\ & \quad \left. + (\mu(A_{4i}^{(k)} \cup A_{4i+2}^{(k)}))^{1/2}\right\}. \end{aligned}$$

The same argument works for $j \in \{1, 3\}$, and consequently

$$\begin{aligned}
 & (\mu(A_{4i+1}^{(k)}))^{1/2} 1_{4i+1 \notin I(k)} + (\mu(A_{4i+3}^{(k)}))^{1/2} 1_{4i+3 \notin I(k)} \\
 (15) \quad & \leq \max\left\{\frac{2^{1/2}}{4}(\mu(A_i^{(k-1)}))^{1/2}, \frac{2^{1/2}}{8}(\mu(A_i^{(k-1)})) \right. \\
 & \quad \left. + (\mu(A_{4i+1}^{(k)} \cup A_{4i+3}^{(k)}))^{1/2}\right\}.
 \end{aligned}$$

Since $x^{1/2} + y^{1/2} \leq 2^{1/2}(x + y)^{1/2}$, for $x, y \geq 0$ we have

$$(16) \quad (\mu(A_{4i}^{(k)} \cup A_{4i+2}^{(k)}))^{1/2} + (\mu(A_{4i+1}^{(k)} \cup A_{4i+3}^{(k)}))^{1/2} \leq 2^{1/2}(\mu(A_i^{(k-1)}))^{1/2}.$$

On the other hand,

$$(17) \quad \max\{(\mu(A_{4i}^{(k)} \cup A_{4i+2}^{(k)}))^{1/2}, (\mu(A_{4i+1}^{(k)} \cup A_{4i+3}^{(k)}))^{1/2}\} \leq (\mu(A_i^{(k-1)}))^{1/2}.$$

By (16) and (17) we obtain that

$$\sum_{j=0}^3 (\mu(A_{4i+j}^{(k)}))^{1/2} 1_{4i+j \notin I(k)} \leq L(\mu(A_i^{(k-1)})),$$

where $L = 2^{1/2} \cdot \frac{5}{4}$. Consequently,

$$(18) \quad \sum_{i=0}^{4^k-1} (\mu(A_i^{(k)}))^{1/2} 1_{i \notin I(k)} \leq L \sum_{i=0}^{4^{k-1}-1} (\mu(A_i^{(k-1)})).$$

Using (18), we deduce

$$\begin{aligned}
 & \sum_{k=1}^{\infty} 2^{-k} \sum_{i=0}^{4^k-1} (\mu(A_i^{(k)}))^{1/2} \\
 & \leq \sum_{k=1}^{\infty} 2^{-k} \sum_{i=0}^{4^k-1} (\mu(A_i^{(k)}))^{1/2} 1_{i \in I(k)} + L \sum_{k=1}^{\infty} 2^{-k} \sum_{i=0}^{4^{k-1}-1} (\mu(A_i^{(k-1)}))^{1/2}.
 \end{aligned}$$

Since $\mu(A_0^{(0)}) = 1$, it implies that

$$(1 - 2^{-1}L) \sum_{k=1}^{\infty} 2^{-k} \sum_{i=0}^{4^k-1} (\mu(A_i^{(k)}))^{1/2} \leq L + \sum_{k=1}^{\infty} 2^{-k} \sum_{i=0}^{4^k-1} (\mu(A_i^{(k)}))^{1/2} 1_{i \in I(k)}.$$

To complete the proof it suffices to apply Lemma 1. \square

4. The partitioning scheme. In this section we prove the main induction procedure. Recall that $(F_k)_{k \geq 0}$ are set functionals defined in (5). We are going to show that

$$(19) \quad \sup_X \left(\mathbf{E} \sup_{t \in T} (X(t) - X(0))^2 \right)^{1/2} \geq \frac{1}{64} \sum_{k=0}^{\infty} 2^{-k} \sum_{j=0}^3 (\mu(A_{4i+j}^{(k)}))^{1/2} 1_{4i+j \in I(k)},$$

where the supremum is taken over all orthogonal processes on T . The idea of the proof is to first show that $F_0(T) < \sup_X (\mathbf{E} \sup_{t \in T} (X(t) - X(0))^2)^{1/2}$. Then we establish the induction step so that $(\mu(A_i^{(k-1)}))^{1/2} F_{k-1}(A_i^{(k-1)})$ can be used to bound $\sum_{j=0}^3 (\mu(A_{4i+j}^{(k)}))^{1/2} F_k(A_{4i+j}^{(k)})$ for all $k \geq 1$ and $0 \leq i < 4^{k-1}$ together with some additional term required to get (19).

First consider the special case of $A_0^{(0)} = T$. For each $Y(t)$, $t \in \bar{A}_0^{(0)}$ satisfying (6) for $k = 0$, we take Z independent of Y such that $\mathbf{E}Z = 0$, $\mathbf{E}Z^2 = 1$. Then the process

$$X(t) = Y(t) + tZ, \quad t \in T,$$

satisfies (1) and, moreover, by Jensen's inequality,

$$(20) \quad \mathbf{E} \sup_{t \in T} Y(t) = \mathbf{E} \sup_{t \in T} (Y(t) - Y(0)) \leq \left(\mathbf{E} \sup_{t \in T} (X(t) - X(0))^2 \right)^{1/2}.$$

Therefore (2) implies that $F_0(T) < \infty$, which makes the induction accessible.

The crucial idea is to show that the induction step is valid.

PROPOSITION 2. *For each $A_i^{(k-1)}$, $0 \leq i < 4^{k-1}$ and $k \geq 1$, the following inequality holds:*

$$\begin{aligned} & (\mu(A_i^{(k-1)}))^{1/2} F_{k-1}(A_i^{(k-1)}) \\ & \geq \frac{1}{64} 2^{-k} \sum_{j=0}^3 (\mu(A_{4i+j}^{(k)}))^{1/2} 1_{4i+j \in I(k)} + \sum_{j=0}^3 (\mu(A_{4i+j}^{(k)}))^{1/2} F_k(A_{4i+j}^{(k)}). \end{aligned}$$

PROOF. Fix $A_i^{(k-1)}$, $0 \leq i < 4^{k-1}$, $k \geq 1$. We may assume that $\mu(A_i^{(k-1)}) > 0$, since otherwise there is nothing to prove. On each $\bar{A}_{4i+j}^{(k)}$, $0 \leq j \leq 3$, there exist a process Y_j , such that

$$\mathbf{E}|Y_l(t) - Y_l(s)|^2 = |t - s|(1 - 4^k|t - s|) \quad \text{for } s, t \in \bar{A}_{4i+j}^{(k)}$$

and

$$(21) \quad \mathbf{E} \sup_{t \in A_{4i+j}^{(k)}} Y_j(t) \geq F_k(A_{4i+j}^{(k)}) - \varepsilon.$$

As we have mentioned, we may assume that $Y_j((4i+j)4^{-k}) = Y_j((4i+j+1)4^{-k}) = 0$. Our goal is to construct a process, $Y(t)$, $t \in \bar{A}_i^{(k-1)}$, using Y_j , $0 \leq j \leq 3$, that verifies (6) for $\bar{A}_i^{(k-1)}$.

To construct $Y(t)$, $t \in T$, we will need also a family of independent random variables Z_j , $0 \leq j \leq 3$. We require that Z_j are independent of processes Y_j , $0 \leq j \leq 3$, and such that $\mathbf{E}Z_j = 0$ and $\mathbf{E}Z_j^2 = 1$. Let $S_0 = 0$ and for $1 \leq j \leq 4$,

$$S_j = \sum_{l=0}^{j-1} Z_l - j4^{-1} \left(\sum_{l=0}^3 Z_l \right) \quad \text{for } 1 \leq j \leq 4.$$

Observe that for $0 \leq l, m \leq 4$,

$$\mathbf{E}|S_l - S_m|^2 = |l - m|(1 - 4^{-1}|l - m|).$$

With the family Z_j , $0 \leq j \leq 3$, we associate a random variable τ valued in $\{0, 1, 2, 3\}$. We require that τ is independent of Y_j , $0 \leq j \leq 3$, and distributed as follows:

$$(22) \quad \mathbf{P}(\tau = j) = \frac{\mu(A_{4i+j}^{(k)})}{\mu(A_i^{(k-1)})} \quad \text{for } 0 \leq j \leq 3.$$

We define the process $Y(t)$, $t \in \bar{A}_{4i+j}^{(k)}$, by

$$(23) \quad \begin{aligned} Y(t) &= 2^{-k}S_j + 2^k(t - (4i+j)4^{-k})(S_{j+1} - S_j) \\ &\quad + (\mathbf{P}(\tau = j))^{-1/2}Y_j(t)1_{\tau=j}, \end{aligned}$$

and also set $Y(i4^{-(k-1)}) = Y((i+1)4^{-(k-1)}) = 0$. We have to show that $Y(t)$, $t \in \bar{A}_i^{(k-1)}$, is admissible for $F_k(A_i^{(k-1)})$, that is, we make thorough calculations for the variance of $Y(s) - Y(t)$, where $s, t \in \bar{A}_i^{(k-1)}$.

LEMMA 2. *The process $Y(t)$, $t \in \bar{A}_i^{(k-1)}$, satisfies $\mathbf{E}Y(t) = 0$, $t \in \bar{A}_i^{(k-1)}$, and*

$$(24) \quad \mathbf{E}|Y(s) - Y(t)|^2 = |s - t|(1 - 4^{k-1}|s - t|) \quad \text{for } s, t \in \bar{A}_i^{(k-1)}.$$

PROOF. The first assertion is trivial; we show (24). Assume that $s, t \in \bar{A}_{4i+j}^{(k)}$, and then by (22), the independence of Z_j , $0 \leq j \leq 3$, and independence between Z_j , $0 \leq j \leq 3$, τ and Y_j , $0 \leq j \leq 3$ [recall that $\mathbf{E}Z_j = 0$ and $\mathbf{E}Y_j(t) = 0$, $t \in \bar{A}_{4i+j}^{(k)}$] we obtain that

$$\begin{aligned} \mathbf{E}|Y(s) - Y(t)|^2 &= 4^k|s - t|^2\mathbf{E}(S_{j+1} - S_j)^2 + \mathbf{P}(\tau = j)\mathbf{P}(\tau = j)^{-1}|s - t|(1 - 4^k|s - t|) \\ &= 4^k(1 - 4^{-1})|t - s|^2 + |s - t|(1 - 4^k|s - t|) = |s - t|(1 - 4^{k-1}|s - t|). \end{aligned}$$

Now suppose that $s \in \bar{A}_{4i+l}^{(k)}$, $t \in \bar{A}_{4i+m}^{(k)}$ and $l < m$. The idea we follow is to rewrite $|Y(s) - Y(t)|^2$ in terms of Z_j , $0 \leq j \leq 3$ and τ . Using that Z_j , $0 \leq j \leq 3$ are independent and Z_j , $0 \leq j \leq 3$, τ are independent of Y_j , $0 \leq j \leq 3$ [moreover $\mathbf{E}Z_j = 0$ and $\mathbf{E}Y_j(t) = 0$, $t \in \bar{A}_{4i+j}$]

$$\begin{aligned} \mathbf{E}(Y(s) - Y(t))^2 &= \mathbf{E}(Y_l(s))^2 + \mathbf{E}(Y_m(t))^2 \\ (25) \quad &+ \mathbf{E}(Y(s) - (\mathbf{P}(\tau = l))^{-1/2}Y_l(s)1_{\tau=l} - Y(t) \\ &+ (\mathbf{P}(\tau = l))^{-1/2}Y_m(t)1_{\tau=m})^2. \end{aligned}$$

Clearly,

$$\begin{aligned} \mathbf{E}(Y_l(s))^2 &= \mathbf{E}(Y_l(s) - Y_l((4i + l + 1)4^{-k}))^2 \\ (26) \quad &= |s - (4i + l + 1)4^{-k}|(1 - 4^{-k}|s - (4i + l + 1)4^{-k}|) \end{aligned}$$

and

$$\begin{aligned} \mathbf{E}(Y_m(t))^2 &= \mathbf{E}(Y_m((4i + m)4^{-k}) - Y_j(t))^2 \\ (27) \quad &= |(4i + m)4^{-k} - t|(1 - 4^{-k}|(4i + m)4^{-k} - t|). \end{aligned}$$

Then we observe that by the definition,

$$Y(s) - (\mathbf{P}(\tau = l))^{-1/2}Y_l(s)1_{\tau=l} = 2^{-k}S_l + 2^k(s - (4i + l)4^{-k})(S_{l+1} - S_l),$$

$$Y(t) - (\mathbf{P}(\tau = m))^{-1/2}Y_l(s)1_{\tau=m} = 2^{-k}S_m + 2^k(t - (4i + m)4^{-k})(S_{m+1} - S_m).$$

Hence

$$\begin{aligned} Y(s) - (\mathbf{P}(\tau = l))^{-1/2}Y_l(s)1_{\tau=l} - Y(t) + (\mathbf{P}(\tau = l))^{-1/2}Y_m(t)1_{\tau=m} \\ = 2^{-k}(S_m - S_l) \\ + 2^k[(t - (4i + m)4^{-k})(S_{m+1} - S_m) - ((s - (4i + l)4^{-k}))(S_{l+1} - S_l)]. \end{aligned}$$

Since $S_j = \sum_{l=0}^{j-1} Z_l - j4^{-1}(\sum_{l=0}^3 Z_l)$, we have

$$\begin{aligned} &2^{-k}(S_m - S_l) \\ &+ 2^k[(t - (4i + m)4^{-k})(S_{m+1} - S_m) - ((s - (4i + l)4^{-k}))(S_{l+1} - S_l)] \\ &= 2^{-k} \left(\sum_{j=l}^{m-1} Z_j - \frac{m-j}{4} \sum_{j=0}^3 Z_j \right) \\ &+ 2^k \left[(t - (4i + m)4^{-k}) \left(Z_m - \frac{1}{4} \sum_{j=0}^3 Z_j \right) \right] \\ &- 2^k \left[((s - (4i + l)4^{-k})) \left(Z_l - \frac{1}{4} \sum_{j=0}^3 Z_j \right) \right]. \end{aligned}$$

We group coefficients by random variables Z_j . For Z_m we obtain

$$\begin{aligned} & -2^{-k} \frac{m-l}{4} + 2^k(t - (4i+m)4^{-k}) \frac{3}{4} + 2^k(s - (4i+l)4^{-k}) \frac{1}{4} \\ & = 2^k(|(4i+m)4^{-k} - t| - 4^{-1}(2^{-k}(m-l) + 2^k|s-t| - 2^{-k}(m-l))) \\ & = 2^k(|(4i+m)4^{-k} - t| - 4^{-1}|s-t|). \end{aligned}$$

Similarly the coefficient for Z_l equals

$$\begin{aligned} & -2^{-k} \frac{m-l}{4} - 2^k(s - (4i+l)4^{-k}) \frac{3}{4} - 2^k(t - (4i+m)4^{-k}) \frac{1}{4} \\ & = 2^k(|(4i+l+1)4^{-k} - s| \\ & \quad - 4^{-1}(2^{-k}(m-l-1) + 2^k|s-t| - 2^{-k}(m-l-1))) \\ & = 2^k(|s - (4i+l+1)4^{-k}| - 4^{-1}|s-t|)2^k. \end{aligned}$$

For $l < j < m$ the coefficient for Z_j is

$$\begin{aligned} & 2^{-k} \left(1 - \frac{m-l}{4} \right) - 2^k(t - (4i+m)4^{-k}) \frac{1}{4} + 2^k(s - (4i+l)4^{-k}) \frac{1}{4} \\ & = 2^k(4^{-k} - 4^{-1}(m-l)4^{-k} - 4^{-1}(|s-t| - (m-l)4^{-k})) \\ & = 2^k(4^{-k} - 4^{-1}|s-t|) \end{aligned}$$

and finally for $j > m$ and $j < l$

$$-2^{-k} \frac{m-l}{4} - 2^k(t - (4i+m)4^{-k}) \frac{1}{4} + 2^k(s - (4i+l)4^{-k}) \frac{1}{4} = -2^k(4^{-1}|s-t|).$$

Consequently we obtain that

$$\begin{aligned} & Y(s) - (\mathbf{P}(\tau=l))^{-1/2} Y_l(s) 1_{\tau=l} - Y(t) + (\mathbf{P}(\tau=l))^{-1/2} Y_m(t) 1_{\tau=m} \\ & = (|(4i+m)4^{-k} - t| - 4^{-1}|s-t|)2^k Z_m \\ & \quad + (|s - (4i+l+1)4^{-k}| - 4^{-1}|s-t|)2^k Z_l \\ & \quad + \sum_{n=l+1}^{m-1} (4^{-k} - 4^{-1}|s-t|)2^k Z_n - 4^{-1}|s-t|2^k \sum_{n<l, n>m} Z_n. \end{aligned}$$

Therefore by the orthogonality of Z_j , $j \in \{0, 1, 2, 3\}$,

$$\begin{aligned} & \mathbf{E}(Y(s) - (\mathbf{P}(\tau=l))^{-1/2} Y_l(s) 1_{\tau=l} - Y(t) + (\mathbf{P}(\tau=l))^{-1/2} Y_m(t) 1_{\tau=m})^2 \\ (28) \quad & = (|(4i+m)4^{-k} - t| - 4^{-1}|s-t|)^2 4^k \\ & \quad + (|s - (4i+l+1)4^{-k}| - 4^{-1}|s-t|)^2 4^k \\ & \quad + (4^{-k} - 4^{-1}|s-t|)^2 (m-l-1)4^k + 4^{-2}|s-t|^2 (4-m+l-1)4^k. \end{aligned}$$

Combining (25), (26), (27), (28) and

$$|s - (4i + l + 1)4^{-k}| + (m - l - 1)4^{-k} + |(4i + m)4^{-k} - t| = |s - t|,$$

we obtain that

$$\mathbf{E}|Y(s) - Y(t)|^2 = |s - t|(1 - 4^{k-1}|s - t|).$$

This completes the proof. \square

Having the process $Y(t)$, $t \in \bar{A}_i^{(k-1)}$, constructed, we use it to provide a lower bound on $F_{k-1}(A_i^{(k-1)})$. First note that

$$F_k(A_i^{(k-1)}) \geq \mathbf{E} \sup_{t \in A_i^{(k-1)}} Y(t) \geq \sum_{j=0}^3 \mathbf{E} \left(\sup_{t \in A_{4i+j}^{(k)}} Y(t) 1_{\tau=j} \right).$$

Moreover,

$$\begin{aligned} & \mathbf{E} \left(\sup_{t \in A_{4i+j}^{(k)}} Y(t) 1_{\tau=j} \right) \\ &= 2^{-k} \mathbf{E} S_j 1_{\tau=j} + \mathbf{E} \left(\sup_{t \in A_{4i+j}^{(k)}} (2^k(t - (4i + j)4^{-k})(S_{j+1} - S_j) 1_{\tau=j} \right. \\ & \quad \left. + (\mathbf{P}(\tau = j))^{-1/2} Y_j(t)) 1_{\tau=j} \right). \end{aligned}$$

Conditioning on $\mathcal{F} = \sigma(Y_j, 0 \leq j \leq 3)$ and then using Jensen's inequality, we deduce

$$\begin{aligned} & \mathbf{E} \sup_{t \in A_{4i+j}^{(k)}} ((2^k(t - (4i + j)))(S_{j+1} - S_j) 1_{\tau=j} + (\mathbf{P}(\tau = j))^{-1/2} Y_j(t)) 1_{\tau=j} \\ & \geq \mathbf{E} \sup_{t \in A_{4i+j}^{(k)}} (\mathbf{E}((2^k(t - (4i + j)4^{-k})(S_{j+1} - S_j) 1_{\tau=j} \\ & \quad + (\mathbf{P}(\tau = j))^{-1/2} Y_j(t)) 1_{\tau=j} | \mathcal{F})) \\ & = \mathbf{E} \sup_{t \in A_{4i+j}^{(k)}} (2^k \mathbf{E}((t - (4i + j)4^{-k})(S_{j+1} - S_j) 1_{\tau=j}) + (\mathbf{P}(\tau = j))^{1/2} Y_j(t)) \\ & \geq -2^{-k} (\mathbf{E}(S_{j+1} - S_j) 1_{\tau=j})_- + (\mathbf{P}(\tau = j))^{1/2} \mathbf{E} \sup_{t \in A_{4i+j}^{(k)}} Y_j(t). \end{aligned}$$

Consequently,

$$\begin{aligned} F_{k-1}(A_i^{(k-1)}) & \geq \sum_{j=0}^3 \left(2^{-k} [\mathbf{E} S_j 1_{\tau=j} - (\mathbf{E}(S_{j+1} - S_j) 1_{\tau=j})_-] \right. \\ & \quad \left. + (\mathbf{P}(\tau = j))^{1/2} \mathbf{E} \sup_{t \in A_{4i+j}^{(k)}} Y_j(t) \right). \end{aligned}$$

Together with (21) and (22) it implies that

$$(29) \quad F_{k-1}(A_i^{(k-1)}) \geq \sum_{j=0}^3 (2^{-k} [\mathbf{E}S_j 1_{\tau=j} - (\mathbf{E}(S_{j+1} - S_j) 1_{\tau=j})_-] + (\mathbf{P}(\tau = j))^{1/2} (F_k(A_{4i+j}^k) - 4\varepsilon).$$

To complete the lower bound, we have to construct variables Z_j , $0 \leq j \leq 3$, and τ . The main idea is to choose $n \in \{0, 1, 2, 3\}$ and variable Z_n to be τ measurable, whereas all remaining Z_j , $j \neq n$, are independent of τ . Therefore we first define τ so that (22) holds, then obtain Z_n as a Borel function of τ and only then set any independent Z_j , $j \neq n$, independent of Z_n . In this setting, define

$$V_n = \sum_{j=0}^3 (\mathbf{E}S_j 1_{\tau=j} - (\mathbf{E}(S_{j+1} - S_j) 1_{\tau=j})_-).$$

Observe that since Z_l , $l \neq n$, are independent of τ and consequently of Z_n , we have $\mathbf{E}Z_l 1_{\tau=j} = \mathbf{E}Z_l \mathbf{P}(\tau = j) = 0$, whenever $l \neq n$. Therefore

$$\mathbf{E}S_j 1_{\tau=j} = \mathbf{E}Z_n 1_{\tau=j} 1_{n \leq j-1} - \frac{j}{4} \mathbf{E}Z_n 1_{\tau=j}$$

and

$$\mathbf{E}(S_{j+1} - S_j) 1_{\tau=j} = \mathbf{E}Z_n 1_{\tau=j} 1_{j=n} - \frac{1}{4} \mathbf{E}Z_n 1_{\tau=j}.$$

Consequently for $j \neq n$, $(\mathbf{E}(S_{j+1} - S_j) 1_{\tau=j})_- = -\frac{1}{4}(\mathbf{E}Z_n 1_{\tau=j})_+$ and for $j = n$, $(\mathbf{E}(S_{j+1} - S_j) 1_{\tau=j})_- = \frac{3}{4}(\mathbf{E}Z_n 1_{\tau=j})_-$. Hence the representation

$$V_n = \sum_{j=n+1}^3 c_j - (1 - 4^{-1})(c_n)_- - \sum_{j=0}^3 j 4^{-1} c_j - \sum_{l \neq n} 4^{-1} (c_j)_+,$$

where $c_j = \mathbf{E}Z_n 1_{\tau=j}$. Since $\varepsilon > 0$ is arbitrary in (29), we obtain

$$(30) \quad F_{k-1}(A_i^{(k-1)}) \geq 2^{-k} V_n + \sum_{j=0}^3 (\mathbf{P}(\tau = j))^{1/2} F_k(A_{4i+j}^{(k)}).$$

The above inequality completes the first part of the proof. Using the process $Y(t)$, $t \in \bar{A}_i^{(k-1)}$, we have shown that $\mu(A_i^{(k-1)}) F_{k-1}(A_i^{(k-1)})$ dominates $\sum_{j=0}^3 \mu(A_{4i+j}^{(k)}) F_k(A_{4i+j}^{(k)})$, together with the additional term $2^{-k} V_n$.

We claim that it is always possible to define Z_n with respect to τ in a way that one can bound V_n from below by a universal constant, assuming that there exists at least one $j \in \{0, 1, 2, 3\}$ such that $4i + j \in I(k)$.

LEMMA 3. *There exists Z_3 measurable with respect to τ , such that $\mathbf{E}Z_3 = 0$, $\mathbf{E}Z_3^2 = 1$ and*

$$(31) \quad V_3 \geq \frac{1}{4} \left(\frac{\mathbf{P}(\tau=0)\mathbf{P}(\tau=2)}{\mathbf{P}(\tau=0) + \mathbf{P}(\tau=2)} \right)^{1/2}$$

and Z_2 measurable with respect to τ , such that $\mathbf{E}Z_2 = 0$, $\mathbf{E}Z_2^2 = 1$ and

$$(32) \quad V_2 \geq \frac{1}{4} \left(\frac{\mathbf{P}(\tau=1)\mathbf{P}(\tau=3)}{\mathbf{P}(\tau=1) + \mathbf{P}(\tau=3)} \right)^{1/2}.$$

PROOF. First note that $\sum_{j=0}^3 c_j = 0$, and then observe that it benefits to set $c_n = 0$. The first case we consider is $n = 3$, so $c_3 = 0$, and then if $c_0 \geq 0$, $c_1 = 0$, $c_2 \leq 0$, we have

$$(33) \quad V_3 = -\frac{1}{4}c_0 - \frac{2}{4}c_2 = -\frac{1}{4}c_2 = \frac{1}{4}c_0,$$

where we have used that $c_0 + c_2 = 0$. The second case is when $n = 2$, $c_2 = 0$, and then if $c_0 = 0$, $c_1 \leq 0$, $c_3 \geq 0$, we have

$$(34) \quad V_2 = c_3 - \frac{1}{4}c_1 - \frac{3}{4}c_3 - \frac{1}{4}c_3 = -\frac{1}{4}c_1 = \frac{1}{4}c_3,$$

where we have used that $c_1 + c_3 = 0$. In the same way one can treat V_0 and V_1 .

The above discussion leads to the definition of Z_n . If $n = 3$, we set

$$Z_3 = x1_{\tau=0} + y1_{\tau=2}.$$

Our requirements are $\mathbf{E}Z_3 = 0$, $\mathbf{E}Z_3^2 = 1$, so

$$\begin{aligned} x\mathbf{P}(\tau=0) + y\mathbf{P}(\tau=2) &= 0, \\ x^2\mathbf{P}(\tau=0) + y^2\mathbf{P}(\tau=2) &= 1. \end{aligned}$$

Therefore

$$x = \left(\frac{\mathbf{P}(\tau=2)}{\mathbf{P}(\tau=0)(\mathbf{P}(\tau=0) + \mathbf{P}(\tau=2))} \right)^{1/2},$$

and consequently all the requirements for (33) are satisfied, and we have

$$V_3 = \frac{1}{4}c_0 = \frac{1}{4} \left(\frac{\mathbf{P}(\tau=0)\mathbf{P}(\tau=2)}{\mathbf{P}(\tau=0) + \mathbf{P}(\tau=2)} \right)^{1/2}.$$

The same argument for $n = 2$ shows that one can construct Z_2 in a way that all requirements for (34) are satisfied and

$$V_2 = \frac{1}{4}c_3 = \frac{1}{4} \left(\frac{\mathbf{P}(\tau=1)\mathbf{P}(\tau=3)}{\mathbf{P}(\tau=1) + \mathbf{P}(\tau=3)} \right)^{1/2}.$$

□

We use the above lemma in (30) to bound $2^{-k}V_n$. There are three cases. First suppose that $4i + j \notin I(k)$ for $0 \leq j \leq 3$, and then we set Z_j , $j \in \{0, 1, 2, 3\}$, to be independent of τ which implies that $V_n = 0$ for any choice of n . Therefore by (30),

$$\begin{aligned} F_{k-1}(A_i^{(k-1)}) &\geq \sum_{j=0}^3 (\mathbf{P}(\tau = j))^{1/2} F_k(A_{4i+j}^{(k)}) \\ &= \frac{1}{64} 2^{-k} \sum_{j=4i}^{4i+3} (\mathbf{P}(\tau = j))^{1/2} 1_{4i+j \in I(k)} \\ &\quad + \sum_{j=0}^3 (\mathbf{P}(\tau = j))^{1/2} F_k(A_{4i+j}^{(k)}). \end{aligned}$$

The second case is that $4i + j \in I(k)$ for $j \in \{0, 2\}$, then we use (8) and (31)

$$\begin{aligned} V_3 &\geq \frac{1}{4} \left(\frac{\mathbf{P}(\tau=0)\mathbf{P}(\tau=2)}{\mathbf{P}(\tau=0) + \mathbf{P}(\tau=2)} \right)^{1/2} \geq \frac{1}{4} \left(\frac{1}{2} \cdot \frac{1}{32} \right)^{1/2} = \frac{1}{32} \\ &\geq \frac{1}{64} \sum_{j=4i}^{4i+3} (\mathbf{P}(\tau = j))^{1/2} 1_{j \in I(k)}, \end{aligned}$$

where we have used the inequality $x^{1/2} + y^{1/2} + z^{1/2} + t^{1/2} \leq 2(x + y + z + t)^{1/2}$, for $x, y, z, t \geq 0$. Therefore

$$\begin{aligned} F_{k-1}(A_i^{(k-1)}) &\geq 2^{-k} V_3 + \sum_{j=0}^3 (\mathbf{P}(\tau = j))^{1/2} F_k(A_{4i+j}^{(k)}) \\ &\geq \frac{1}{64} 2^{-k} \sum_{j=4i}^{4i+3} (\mathbf{P}(\tau = j))^{1/2} 1_{4i+j \in I(k)} \\ &\quad + \sum_{j=0}^3 (\mathbf{P}(\tau = j))^{1/2} F_k(A_{4i+j}^{(k)}). \end{aligned}$$

The third possibility is that $4i + j \in I(k)$, $j \in \{1, 3\}$, and then by (9) and (32) we have

$$\begin{aligned} V_2 &\geq \frac{1}{4} \left(\frac{\mathbf{P}(\tau=1)\mathbf{P}(\tau=3)}{\mathbf{P}(\tau=1) + \mathbf{P}(\tau=3)} \right)^{1/2} \geq \frac{1}{4} \left(\frac{1}{2} \cdot \frac{1}{32} \right)^{1/2} = \frac{1}{32} \\ &\geq \frac{1}{64} \sum_{j=4i}^{4i+3} (\mathbf{P}(\tau = j))^{1/2} 1_{4i+j \in I(k)}. \end{aligned}$$

Consequently

$$\begin{aligned}
F_{k-1}(A_i^{(k-1)}) &\geq 2^{-k}V_3 + \sum_{j=0}^3 (\mathbf{P}(\tau = j))^{1/2} F_k(A_{4i+j}^{(k)}) \\
&\geq \frac{1}{64} 2^{-k} \sum_{j=4i}^{4i+3} (\mathbf{P}(\tau = j))^{1/2} 1_{4i+j \in I(k)} \\
&\quad + \sum_{j=0}^3 (\mathbf{P}(\tau = j))^{1/2} F_k(A_{4i+j}^{(k)}).
\end{aligned}$$

In the view of (22) it completes the proof of Proposition 2. \square

5. Proof of the main result. In this section we use the functional F_k , $k \geq 0$, and the induction scheme proved in Proposition 2 to prove (19).

PROPOSITION 3. *The following inequality holds:*

$$\sum_{k=1}^{\infty} 2^{-k} \sum_{i=0}^{4^k-1} (\mu(A_i^{(k)}))^{1/2} 1_{i \in I(k)} \leq 64 \left(\sup_X \mathbf{E} \sup_{t \in T} (X(t) - X(0))^2 \right)^{1/2},$$

where the supremum is taken over all orthogonal process on T .

PROOF. By (20) we have

$$F_0(T) \leq \left(\sup_X \mathbf{E} \sup_{t \in T} (X(t) - X(0))^2 \right)^{1/2}.$$

On the other hand using the induction step proved in Proposition 2, we deduce

$$\sum_{k=1}^{\infty} 2^{-k} \sum_{i=0}^{4^k-1} (\mu(A_i^{(k)}))^{1/2} 1_{i \in I(k)} \leq 64 F_0(T).$$

This completes the proof. \square

Using Propositions 1 and 2, we conclude Theorem 4 with

$$K = \frac{1}{1 - 2^{-1}L} \left(L + 64 \sup_X \left(\mathbf{E} \sup_{t \in T} (X(t) - X(0))^2 \right)^{1/2} \right),$$

and $L = 2^{1/2} \cdot \frac{5}{4}$.

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